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On the inverse scattering of star-shape LC-networks

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***Abstract*— The study of the scattering data for a star-shape network of LC-transmission lines is transformed into the scattering analysis of a Schrödinger operator on the same graph. The boundary conditions coming from the Kirchhoff rules ensure the existence of a unique self-adjoint extension of the mentioned Schrödinger operator. While the graph consists of a number of infinite branches and a number finite ones, all joining at a central node, we provide a construction of the scattering solutions. Under non-degenerate circumstances (different wave travelling times for finite branches), we show that the study of the reflection coefficient in the high-frequency regime must provide us with the number of the infinite branches as well as the the wave travelling times for finite ones.**

I. Introduction

The number of electronic equipments is increasing rapidly in automotive vehicles, aircrafts, and many other safety critical systems. Consequently, the reliability of wired networks and electric connections is becoming more and more important. For example, in automotive industry, a goal is to develop compact and easy to use devices for the diagnosis of electric connection failures in garage or at the end of the production chain. These devices should be capable of detecting and locating failures in cables and in connectors. Another goal is on-board diagnosis: the diagnosis device will be integrated to the vehicle in order to detect failures under normal working conditions of the vehicle. To find faulty wiring in such networks, it is not always possible to measure end-to-end cable impedances, because the number of available diagnostic port plugs is limited, and furthermore, for diagnosis purpose, it is not sufficient to detect a high end-to-end impedance as it is also necessary to locate the fault within the cable. In such situations Time or Frequency Domain Reflectometry (TDR, FDR) are the most commonly used methods: a high frequency signal (a short travelling pulse for TDR, a standing wave for FDR) is sent down a wire at some point and the signal reflected by the network is measured at the same point and analyzed for fault detection and location [11], [16], [5], [2]. Automatic fault detection and diagnosis using reflectometry methods is the subject of intense research, both on the technologies of ”smart wiring systems” and reflectometers [15] and on the foundations of the TDR/FDR methods. Most studies are concerned with various aspects of mathematical modeling and simulation of microwave propagation in networks, from network complexity [12] to reduced order modeling of high frequency phenomena like skin effect [1]. The inverse problem of fault detection and localization for a network is in general studied in simulation or experimentally. For this problem, the available theory is developed mainly in the framework of Inverse Scattering Theory limited to very simple networks (segment, half-line, line) (see e.g. [3], [13]).

In this paper, we consider the inverse scattering problem (ISP) for a star-shape network Γ of transmission lines. Some of the branches of the network are extended in a direction z to $+\infty$ and the others admit a finite length. The network is assumed to be non-dissipative; i.e. the series resistance $R(z)$ and the shunt conductance $G(z)$ per unit length are assumed to vanish and therefore, we are dealing with LC-wires.

The main concern here is to derive some information concerning the topology and the geometry of the network, applying a very few scattering information. Indeed, fixing an infinite branch e_1 (see Figure 1), we generate a high-frequency time varying voltage at its infinite end and measuring the intensity at the same end, we would like to derive information such as the number of infinite branches m and the wave travelling time τ_{m+j} of the finite segments e_{m+j}.

Here, in order to model our network, each branch is parameterized through its own length. The central node is assumed to be the zero (origin) of all the branches. While the position coordinate z_j takes values inside the interval $[0,l_j]$, l_j being the length of the branch e_j and therefore

$$l_j = \infty \quad \forall j \in \{1,\cdots,m\} \quad \text{and} \quad l_j < \infty \quad \forall j \in \{m+1,\cdots,m+n\}.$$

Moreover, on each branch, of finite or infinite length, we suppose that

- A1 $L(z)$ the inductance and $C(z)$ the capacitance are sufficiently regular (twice differentiable);
- A2 $L(z) > 0$ and $C(z) > 0$.
- A3 On infinite branches, $L(z)$ and $C(z)$ have strictly positive finite limits $L(\infty)$ and $C(\infty)$ as $z \to \infty$.

In the next section, we provide the mathematical model consisting of the transmission line equations on each one of the network branches and the boundary conditions (Kirchhoff rules) coupling these equations together. Next, for the

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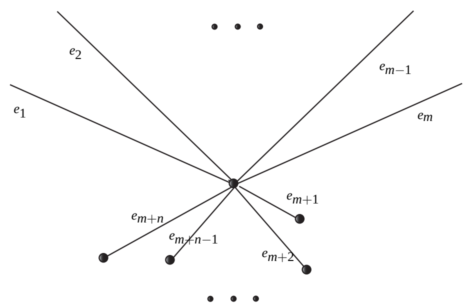

Fig. 1. The *LC*-network Γ consisting of m infinite branches extended to $z=\infty$ and n finite branches of various lengthes.

case of a simple transmission line, we provide a brief review of an old result by I. Kay [9], transforming the inverse scattering problem for one-dimensional nonuniform non dissipative transmission lines to the inverse scattering problem for a Schrödinger operator. We extend this result to the case of our network by defining relevant boundary conditions. Finally, we provide a result ensuring the essential self-adjointness of the Schrödinger operator on the graph with the aforementioned boundary conditions. The existence of a unique self-adjoint extension allows us to start the study of the scattering theory and the scattering solutions.

Next, in Section III, we solve the scattering problem by providing the scattering solutions. In this aim, we apply the Jost solutions on semi-infinite branches [7], [6] and the fundamental solutions on the finite segments [10].

Finally, in section IV, we study the asymptotic behavior of the reflection coefficient in the limit of high-frequency waves and we prove how this limit provides us the needed information on the topology and the geometry of the graph.

II. Mathematical model

A. Transmission line equations

In the absence of dissipation, the transmission line equations writes

$$\begin{cases} \frac{\partial I}{\partial z}+C(z)\frac{\partial U}{\partial t}=0 \\ \frac{\partial U}{\partial z}+L(z)\frac{\partial I}{\partial t}=0 \end{cases} \tag{1}$$

where $I(z,t)$ and $U(z,t)$, respectively denote the intensity of the current and the voltage at time t and position z of a finite or infinite branch. Let us index these functions as $\{L_j\}_{j=1}^{n+m}$, $\{C_j\}_{j=1}^{n+m}$, $\{I_j\}_{j=1}^{n+m}$ and $\{U_j\}_{j=1}^{n+m}$, defined on the branches $\{e_j\}_{j=1}^{n+m}$.

At the central node, the Kirchhoff rules imply

$$\begin{aligned} &U_i(0)=U_j(0), \qquad i,j\in\{1,\cdots,m+n\} \\ &\sum_{j=1}^{m+n} I_j(0)=0. \end{aligned} \tag{2}$$

At the terminal nodes l_{m+j} of the finite branch e_{m+j}, we assume the boundary condition

$$I_{m+j}(l_{m+j})=0 \qquad j\in\{1,\cdots,n\}. \tag{3}$$

B. Liouville transformation

Here, we follow an old result by I. Kay [9] and extended by M. Jaulent to the dissipative case [8].

For a wave of frequency k, i.e. for

$$\begin{aligned} I(z,t)&=I(k,z)e^{-\imath kt} \\ U(z,t)&=U(k,z)e^{-\imath kt}, \end{aligned}$$

the equation (1) becomes

$$\frac{d}{dz}\left(\frac{1}{L(z)}\frac{dU}{dz}\right)+k^2C(z)U=0 \tag{4}$$

Applying the Liouville transforation

$$x(z)=\int_0^z (L(u)C(u))^{\frac{1}{2}}\,du$$

and the conventions $I(k,z(x))=I(k,x)$, $U(k,z(x))=U(k,x)$, etc, and setting

$$y(k,x)=\left[\frac{C(x)}{L(x)}\right]^{\frac{1}{4}}U(k,x) \tag{5}$$

the equation (4) writes

$$\frac{d^2y}{dx^2}+(k^2-V(x))y=0.$$

Here, the potential $V(x)$ is defined as follows:

$$V(x)=\left[\frac{C(x)}{L(x)}\right]^{-\frac{1}{4}}\frac{d^2}{dx^2}\left[\frac{C(x)}{L(x)}\right]^{\frac{1}{4}}. \tag{6}$$

Note that, the new coordinate x here has the dimension of time. It actually parameterizes the travelling time of the wave through the transmission line. In particular the length l of a finite transmission line is transformed into the correspoindg wave travelling time:

$$\tau=\int_0^l (L(z)C(z))^{1/2}dz.$$

C. Extension to the network

The *LC*-transmission line equations defined on each edge for $j\in\{1,\ldots,n+m\}$ are equivalent to the Schrödinger equations

$$\frac{d^2y_j}{dx^2}+(k^2-V_j(x))y_j=0. \tag{7}$$

In this new formalism, the boundary condition (2) writes

$$\begin{aligned} &A_j^{-1}y_j\Big|_{x=0}=A_i^{-1}y_i\Big|_{x=0}=\overline{y} \qquad i,j\in\{1,\cdots,m+n\}, \\ &\sum_{j=1}^{m+n}A_jy_j'\Big|_{x=0}=\sum_{j=1}^{m+n}A_j'y_j\Big|_{x=0}. \end{aligned} \tag{8}$$

Here, applying the assumptions of the Section I, the functions

$$A_j(x_j)=\left[\frac{C_j(x_j)}{L_j(x_j)}\right]^{\frac{1}{4}}, \qquad j\in\{1,\cdots,n+m\}$$

are well defined, strictly positive and regular.

Furthermore, the boundary conditions (3) write

$$y'_{m+j}(\tau_{m+j}) = \frac{A'_{m+j}}{A_{m+j}}\Big|_{x=\tau_{m+j}} y_{m+j}(\tau_{m+j}) \qquad j \in \{1,\cdots,n\}, \quad (9)$$

where $\{\tau_{m+j}\}_{j=1}^{n}$ denote the wave travelling time corresponding the finite transmission lines.

In this paper, we denote

$$h_{m+j} := \frac{A'_{m+j}}{A_{m+j}}\Big|_{x=\tau_{m+j}} \qquad j = 1,\cdots,n. \qquad (10)$$

In conclusion, in order to study the LC-transmission line equations on the graph Γ with boundary conditions (2) and (3), we can study the Schrödinger equation (7) with boundary conditions (8) and (9).

D. Self-adjointness

We are dealing here with a symmetric operator $\mathscr{L}$ defined on $L^2(\Gamma)$:

$$\mathscr{L} = \oplus \mathscr{L}_j, \qquad \mathscr{L}_j = -\frac{d^2}{dx_j^2} + V(x_j) \text{ on } e_j, \quad j = 1,\cdots,m+n,$$

with

$$D(\mathscr{L}) = \text{closure of } C^{\infty}_{\text{b.c.}} \text{ in } H^2(\Gamma).$$

Here, $C^{\infty}_{\text{b.c.}}$ denotes the space of infinitely differentiable functions satisfying the boundary conditions (8) and (9).

In order to start the study of the scattering theory for this operator, we need first to prove that it admits a self-adjoint extension. In this aim, we assume an appropriate decay assumption on $V(x_j)$ ensuring that the operator $\mathscr{L}$ is a compact perturbation of the operator $\oplus_{j=1}^{m+n}\left(-\frac{d^2}{dx_j^2}\right)$. Indeed, one also need to add a new assumption **A4** to the three assumption **A1** through **A3** of the introduction:

A4 On each branch e_j, $j = 1,\cdots,m+n$, we have

$$\int_0^{\tau_j} (1+x_j) \Big| \left[\frac{C_j(x_j)}{L_j(x_j)}\right]^{\frac{1}{4}} \frac{d^2}{dx_j^2} \left[\frac{C_j(x_j)}{L_j(x_j)}\right]^{-\frac{1}{4}} \Big| < \infty.$$

Now, we apply a general result by Carlson [4] on the self-adjointness of differential operators on graphs.

Indeed, following the Theorem 3.4 of [4], we only need to show that at a node connecting N edges, we have N linearly independent linear boundary conditions. At the terminal edges of $\{e_j\}_{j=m+1}^{m+n}$ this is trivially the case as there is one branch and one boundary condition. At the central edge it is not hard to verify that (8) and (9) define such boundary conditions as well. This implies that the operator $\mathscr{L}$ is essentially self-adjoint and therefore that it admits a unique self-adjoint extension. We are now ready to start the study of the scattering solutions.

III. Scattering Solutions

We are interested in the scattering solution where a signal of frequency k is applied at one end of one of the infinite branches. In fact, in general, one might be interested in a experiment where signals of different amplitudes $\{\alpha_j\}_{j=1}^{m}$ are applied at all ends of the infinite branches. In such a case, we will be seeking a scattering solution satisfying the asymptotic behavior at $k \to \infty$

$$y_j(x_j,k) \sim \alpha_j e^{-\imath k x_j} + R_j(k) e^{\imath k x_j}, \qquad j = 1,\cdots,m.$$

However, the experiment we consider here corresponds to the simple case

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_j = 0, \qquad j = 2,\cdots,m.$$

The main result of this section, may be expressed as follows:

Theorem 1: There exists a unique solution

$$\Psi(x,k) = (y_j(x_j,k))_{j=1}^{m+n},$$

of the scattering problem, satisfying

- $\mathscr{L}_j y_j(x_j,k) = k^2 y_j(x_j,k)$;
- $(y_j(x_j,k))_{j=1}^{m+n}$ satisfy the boundary conditions (8) and (9);
- For each $k \in \mathbb{R}$, there exist $R_1(k)$ and $\{T_j(k)\}_{j=2}^{k}$ such that

$$y_1(x_1,k) \sim e^{-\imath k x_1} + R_1(k) e^{\imath k x_1}, \quad x_1 \to \infty, \quad (11)$$
$$y_j(x_j,k) \sim T_j(k) e^{\imath k x_j}, \quad x_j \to \infty, \quad j = 2,\cdots,m. \quad (12)$$

The coefficients, $R_1(k)$ and $\{T_j\}_{j=2}^{m}$, in (11) and (12) appear to be unique. They are called respectively the reflection and the transmission coefficients.

We proceed the proof of the Theorem 1 by constructing such a solution and by showing that such a construction is unique.

In this aim, we apply two different types of solutions: 1- the so-called Jost solutions on half-lines [7], [6] and 2- the fundamental solution of the one-end Sturm-Liouville equation [10].

A. Jost solutions

The Jost solutions $f(x,k;V)$ and $\tilde{f}(x,k;V)$ on the half line $[0,\infty)$, where V is a potential defined on $[0,\infty)$, are the solutions of the Schrödinger integral equations

$$f(x,k;V) = e^{\imath k x} - \int_x^{\infty} \frac{\sin k(x-y)}{k} V(y) f(y,k;V) dy,$$
$$\tilde{f}(x,k;V) = e^{-\imath k x} + \int_0^{x} \frac{\sin k(x-y)}{k} V(y) \tilde{f}(y,k;V) dy. \quad (13)$$

Under the decay assumption A4 on V, these Volterra type integral equations admit unique solutions. In fact, one only needs to apply the standard iterative methods for the proof of existence and uniqueness of solutions to Volterra type integral equations (see e.g. [14], pages 138-139, for a proof).

The functions $f(x,k;V)$ and $\tilde{f}(x,k;V)$ satisfy the the Schrödinger equation

$$-\frac{d^2}{dx^2}f+V(x)f=k^2f,\qquad -\frac{d^2}{dx^2}\tilde{f}+V(x)\tilde{f}=k^2\tilde{f}$$

and admit the asymptotic behaviors (at $x\to\infty$)

$$\begin{aligned}&f(x,k;V)=e^{\imath kx}+o(1),\\ &\tilde{f}(x,k;V)=a(k;V)e^{-\imath kx}+b(k;V)e^{\imath kx}+o(1),\end{aligned}\tag{14}$$

where

$$a(k;V)=1-\frac{1}{2\imath k}\int_0^\infty e^{\imath kx}V(x)\tilde{f}(x,k;V)dx,$$
$$b(k;V)=\frac{1}{2\imath k}\int_0^\infty e^{-\imath kx}V(x)\tilde{f}(x,k;V)dx.$$

Moreover, one can show that ([14], pages 138-139)

$$\left|\frac{d}{dx}f(x,k;V)-\imath ke^{\imath kx}\right|\leq|k|\left|\exp\Big(\frac{1}{k}\int_x^\infty|V(y)|dy\Big)-1\right|.\tag{15}$$

B. Fundamental solution

The fundamental solution, $\omega(x,k;h,V)$, is a solution of the Schrödinger equation,

$$\begin{aligned}&-\frac{d^2}{dx^2}\omega+V(x)\omega=k^2\omega,\qquad x\in(0,\infty),\\ &\omega(0,k;h,V)=1\quad,\quad \omega'(0,k;h,V)=h,\end{aligned}\tag{16}$$

where ω' denotes the derivative with respect to x.

This solution may be expressed through the following integral representation [10],

$$\begin{aligned}\omega(x,k;h,V)&=\cos(kx)+h\frac{\sin(kx)}{k}\\ &+\int_{-x}^{x}K(x,t;V)\{\cos(kt)+h\frac{\sin(kt)}{k}\}dt.\end{aligned}\tag{17}$$

Here the kernel $K(x,t;V)$ is a solution of the following integral equation

$$\begin{aligned}K(x,t;V)&=\frac{1}{2}\int_0^{\frac{x+t}{2}}V(u)du\\ &+\frac{1}{2}\int_0^{\frac{x+t}{2}}d\alpha\int_0^{\frac{x-t}{2}}V(\alpha+\beta)K(\alpha+\beta,\alpha-\beta)d\beta,\end{aligned}\tag{18}$$

where we have extended the potential V to be zero outside the interval of its definition.

The integral equation (18) admits a unique solution (cf. [10], Theorem 1.2.2). Applying the same result, for a potential V in $L^1(\mathbb{R})$, this solution satisfies

$$|K(x,t;V)|\leq\frac{1}{2}\|V\|_{L^1}\exp(x\|V\|_{L^1}),\quad\text{for } x\geq0 \text{ and } |t|\leq x.\tag{19}$$

Furthermore, simple computations imply

$$\begin{aligned}K_x(x,t;V)&=\frac{1}{4}V(\frac{x+t}{2})\\ &+\frac{1}{4}\int_0^{\frac{x-t}{2}}d\beta\ V(\beta+\frac{x+t}{2})K(\frac{x+t}{2}+\beta,\frac{x+t}{2}-\beta)\\ &+\frac{1}{4}\int_0^{\frac{x+t}{2}}d\alpha\ V(\alpha+\frac{x-t}{2})K(\alpha+\frac{x-t}{2},\alpha-\frac{x-t}{2}),\end{aligned}$$

and

$$\begin{aligned}K_t(x,t;V)&=\frac{1}{4}V(\frac{x+t}{2})\\ &+\frac{1}{4}\int_0^{\frac{x-t}{2}}d\beta\ V(\beta+\frac{x+t}{2})K(\frac{x+t}{2}+\beta,\frac{x+t}{2}-\beta)\\ &-\frac{1}{4}\int_0^{\frac{x+t}{2}}d\alpha\ V(\alpha+\frac{x-t}{2})K(\alpha+\frac{x-t}{2},\alpha-\frac{x-t}{2}),\end{aligned}$$

where K_x and K_t denote respectively the derivatives with respect to the first and the second coordinates x and t. For a potential $V\in L^1(\mathbb{R})$, this together with (19) implies

$$\begin{aligned}&|K_x(x,t;V)|,|K_t(x,t;V)|\leq\frac{1}{4}|V(\frac{x+t}{2})|\\ &+\frac{1}{2}\|V\|_{L^1}^2\exp(x\|V\|_{L^1}),\quad\text{for } x\geq0 \text{ and } |t|\leq x.\end{aligned}\tag{20}$$

C. Proof of Theorem 1: constructing the scattering solution

For this construction, we distinguish between three types of edges.

1) Edge e_1: Define

$$\zeta(x_1,k)=y_1(x_1,k)-\frac{1}{a(k;V_1)}\tilde{f}(x_1,k;V_1).$$

Note that

$$a(k;V)=1+\frac{1}{2\imath k}\int_0^\infty V(x)dx+o(1/|k|),\tag{21}$$

and therefore, at least for k large enough, $a(k,V_1)$ is non-zero and $\zeta(x_1,k)$ is well-defined. Furthermore, note that $\zeta(x_1,k)$ satisfies the Schrödinger equation

$$-\frac{d^2}{dx_1^2}\zeta+V_1(x_1)\zeta=k^2\zeta,$$

and through the requirement (11) and by (14), at $x_1\to\infty$

$$\zeta(x_1,k)\sim\left(R_1(k)t-\frac{b(k;V_1)}{a(k;V_1)}\right)e^{\imath kx_1}.\tag{22}$$

Consider now the Wronskian

$$\begin{aligned}&W(\zeta(.,k),\tilde{f}(.,k;V_1))(x_1)=\\ &\qquad\zeta(x_1,k)\tilde{f}'(x_1,k;V_1)-\zeta'(x_1,k)\tilde{f}(x_1,k;V_1),\end{aligned}$$

where the derivatives are taken with respect to the position x_1. It is not a hard task to verify that the above Wronskian is a constant function of x_1. In fact, as the both functions $\zeta(.,k)$ and $\tilde{f}(.,k;V_1)$ verify the same Schrödinger equation with the same potential V_1, one can easily check that

$$\frac{d}{dx_1}W(\zeta(.,k),\tilde{f}(.,k;V_1))(x_1)=0.$$

Applying (22) and (15), we have

$$\lim_{x_1\to\infty}W(\zeta(.,k),\tilde{f}(.,k;V_1))(x_1)=0$$

and therefore

$$W(\zeta(.,k),\tilde{f}(.,k;V_1))(x_1)=0\qquad\text{for } x_1\in[0,\infty).$$

This implies that $\zeta(.,k)$ and $f(.,k;V_1)$ are co-linear and thus for some $R_1(k)$

$$y_1(x_1,k) = \frac{1}{a(k;V_1)}\tilde{f}(x_1,k;V_1) + (R_1(k) - \frac{b(k;V_1)}{a(k;V_1)})f(x_1,k;V_1). \quad (23)$$

2) Edges e_j, $j = 2,\cdots,m$: We consider the Wronskian $W(y_j(.,k), f(.,k;V_j))(x_j)$. Just as in above, this Wronskian is a constant of x_j and applying (12) together with (15),

$$\lim_{x_j\to\infty} W(y_j(.,k),\tilde{f}(.,k;V_j))(x_j) = 0$$

and thus for some $T_j(k)$

$$y_j(x_j,k) = T_j(k)f(x_j,k;V_j), \qquad j = 2,\cdots,m. \quad (24)$$

3) Edges e_j, $j = m+1,\cdots,m+n$: We consider the Wronskian

$$W(y_j(\tau_j - .,k), \omega(\tau_j - .,k;h_j,V_j(\tau_j - .)))(x_j),$$

where h_j is defined in (10). On the segment $[0,\tau_j]$, the functions $y_j(\tau_j - .,k)$ and $\omega(\tau_j - .,k;h_j,V_j(\tau_j - .))$ satisfy the same Schrödinger equation with the same potential $V_j(\tau_j - .)$, and therefore the Wronskian is a constant.

Applying (9) together with the definition of the fundamental solution (16), we have

$$W(y_j(.,k),\omega(\tau_j - .,k;h_j,V_j(\tau_j - .)))\Big|_{x_j=\tau_j} = 0,$$

and therefore, just as in above there exists a constant $\alpha_j(k)$ such that

$$y_j(x_j,k) = \alpha_j(k)\omega(\tau_j - x_j,k;h_j,V_j(\tau_j - .)), \qquad j = m+1,\cdots,m+n. \quad (25)$$

At this point, we can apply the boundary conditions at the centeral node (8), in order to find the $m+n$ constants $R_1(k)$, $\{T_j(k)\}_{j=2}^{m}$ and $\{\alpha_j(k)\}_{j=m+1}^{m+n}$. In fact, we have

$$\begin{aligned}\bar{y} &= A_1^{-1}(0)\left(\frac{1}{a(k;V_1)} + (R_1(k) - \frac{b(k;V_1)}{a(k;V_1)})f(0,k;V_1)\right)\\ &= A_{j_1}^{-1}(0)T_{j_1}(k)f(0,k;V_{j_1})\\ &= A_{j_2}^{-1}(0)\alpha_{j_2}(k)\omega(\tau_j,k;h_j,V_{j_2}(\tau_j - .)),\end{aligned} \quad (26)$$

for $j_1 = 2,\cdots,m$ and $j_2 = m+1,\cdots,m+n$. Here, for the first equality, we have applied $\tilde{f}(0,k;V_1) = 1$. Moreover, we have

$$\begin{aligned}&A_1(0)(\frac{-\imath k}{a(k;V_1)} + (R_1(k) - \frac{b(k;V_1)}{a(k;V_1)})f'(0,k;V_1)) + \sum_{j=2}^{m} A_j(0)T_j(k)f'(0,k;V_j)\\ &+ \sum_{j=m+1}^{m+n} A_j(0)\alpha_j(k)\omega'(\tau_j,k;h_j,V_j(\tau_j - .)) = \bar{y}\sum_{j=1}^{m+n} A_j(0)A_j'(0).\end{aligned} \quad (27)$$

Here, for the first line we have applied $\tilde{f}'(0,k;V) = \imath k$. One can easily check that, (26) and (27) provide $m+n$ linearly independent linear equations for the $m+n$ unknown constants $R_1(k)$, $\{T_j(k)\}_{k=2}^{m}$ and $\{\alpha_j\}_{k=m+1}^{n}$. One can therefore find these coefficients uniquely and solve the scattering problem. □

Here, we are only interested in the high frequency ($k \to \infty$) behavior of the reflection coefficient $R_1(k)$ and we do not need to compute all the coefficients explicitly.

IV. High-frequency asymptotic behavior of reflection coefficient

In this Section, we announce and prove the main result of this paper:

Theorem 2: Consider the assumptions **A1** through **A4**, together with the new one,

A5 The function L/C is continous at the central node od Γ; i.e.

$$\frac{L_{j_1}(0)}{C_{j_1}(0)} = \frac{L_{j_2}(0)}{C_{j_2}(0)}, \qquad j_1, j_2 = 1,\cdots,m+n.$$

Then, in the high-frequency regime ($k \to \infty$) the reflection coefficient $R_1(k)$ satisfy:

$$R_1(k) \sim -\frac{(m-2)\imath - \sum_{j=m+1}^{m+n}\tan(k\tau_j)}{m\imath - \sum_{j=m+1}^{m+n}\tan(k\tau_j)} + o(1). \quad (28)$$

Note that, applying this result one can easily identify m the number of the infinite branches, as well as $\{\tau_j\}_{j=m+1}^{m+n}$ the wave travelling times of the finite ones (when they are all different) through the reflection coefficient $R_1(k)$. Indeed, through simple computations, we have the main result of this paper:

Corollary 1: Under the assumptions **A1** through **A5**, and in the high frequency regime $k \to \infty$, we have

$$m = 2\Re(\frac{1}{1+R_1(k)}) + o(1), \quad (29)$$

and

$$\sum_{j=m+1}^{m+n}\tan(k\tau_j) = 2\Im(\frac{1}{1+R_1(k)}) + o(1), \quad (30)$$

where $\Re$ and $\Im$ denote respectively the real and imaginary parts of a complex number. In particular, scanning a certain interval of frequencies k for $k > k^*$ large enough, and through the study of the poles of $\Im(\frac{1}{1+R(k)})$, we can identify the wave travelling times τ_j.

Before starting the proof of the Theorem 2, we need to prove two lemmas.

Lemma 1: Under the assumption **A4** on $V(x)$, we have

$$\frac{f'(0,k;V)}{f(0,k;V)} \sim \imath k + O(1) \qquad \text{as } k \to \infty. \quad (31)$$

Proof of Lemma 1: Here, we just need to apply the the following approximations on $f(x,k;V)$ and $f'(x,k;V)$ (cf. [14], pages 138-139, for a proof):

$$\begin{aligned}|f(x,k;V) - e^{\imath kx}| &\le \Big|\exp(\frac{1}{|k|}\int_x^\infty |V(s)|ds) - 1\Big|,\\ |\frac{df(x,k;V)}{dx} - \imath k e^{\imath kx}| &\le |k|\Big|\exp(\frac{1}{|k|}\int_x^\infty |V(s)|ds) - 1\Big|.\end{aligned} \quad (32)$$

Applying the integrability of $V(x)$, the relation (31) can be obtained very easily. $\square$

Lemma 2: Under the assumption **A4** on $V(x)$ and for a fixed position $x=\tau$, we have

$$\frac{\omega'(\tau,k;h,V)}{\omega(\tau,k;h,V)} \sim -k\tan(k\tau)+o(1) \qquad \text{as } k\to\infty. \qquad (33)$$

Proof of Lemma 2: We apply here the expression (17) of $\omega(x,k;h,V)$. Through the inequality (19), we now that for a fixed τ, the kernel $K(\tau,t;V)$ is bounded and therefore we easily have

$$\omega(\tau,k;h,V)=\cos(k\tau)+\int_{-\tau}^{\tau}K(\tau,t;V)\cos(kt)dt+O(1/k).$$

developing the integral by parts, we have

$$\int_{-\tau}^{\tau}K(\tau,t;V)\cos(kt)dt=\frac{1}{k}\sin(kt)K(\tau,t;V)\Big|_{t=-\tau}^{t=\tau} - \frac{1}{k}\int_{-\tau}^{\tau}\sin(kt)K_t(\tau,t;V)dt$$

Now, applying the inequality (20), we have

$$\int_{-\tau}^{\tau}K(\tau,t;V)\cos(kt)dt\le\frac{1}{k}\sin(kt)K(\tau,t;V)\Big|_{t=-\tau}^{t=\tau} - \frac{1}{4k}\int_{-\tau}^{\tau}|V(\frac{x+t}{2})|dt+\frac{c}{k}\int_{-\tau}^{\tau}|\sin(kt)|dt\le O(\frac{1}{k}),$$

where we have applied the boundedness of $K(\tau,t;V)$ for $|t|\le\tau$ and the integrability of V. This implies

$$\omega(\tau,k;h,V)=\cos(k\tau)+O(1/k), \qquad \text{as } k\to\infty. \qquad (34)$$

Furthermore, we have

$$\begin{aligned}\omega_x(\tau,k;h,V) &= -k\sin(k\tau)+h\cos(k\tau)\\ &+(K(\tau,\tau)+K(\tau,-\tau))\cos(k\tau)+h\left(K(\tau,\tau)-K(\tau,-\tau)\right)\frac{\sin(kt)}{k}\\ &+\int_{-\tau}^{\tau}K_x(\tau,t;V)\{\cos(kt)+h\frac{\sin(kt)}{k}\}dt.\end{aligned}$$

Similar computations as in above, together with the inequality (20), imply

$$\omega_x(\tau,k;h,V)=-k\sin(k\tau)+O(1), \qquad \text{as } k\to\infty. \qquad (35)$$

The two relations (34) and (35) finish the proof of Lemma 2 and we have (33). $\square$

We are now ready to prove the Theorem 2.

Proof of Theorem 2:

The assumption **A5** implies the existence of a constant $\bar{A}$, such that

$$A_j(0)=\bar{A}, \qquad \forall j=1,\cdots,m+n.$$

Dividing (27) by $\bar{y}$ and applying (26), we have

$$\sum_{j=1}^{n+m}A'_j(0)/\bar{A}=\frac{-\imath k+(a(k;V_1)R_1(k)-b(k;V_1))f'(0,k;V_1)}{1+(a(k;V_1)R_1(k)-b(k;V_1))f(0,k;V_1)} + \sum_{j=2}^{m}\frac{f'(0,k;V_j)}{f(0,k;V_j)}+\sum_{j=m+1}^{m+n}\frac{\omega'(\tau_j,k;h_j,V_j(\tau_j-.))}{\omega(\tau_j,k;h_j,V_j(\tau_j-.))}.$$

Through this relation we can derive the value of the reflection coefficient $R_1(k)$. The high-frequency behavior of this reflection coefficient may be derived through the Lemmas 1 and 2 and the relations (21) and

$$b(k;V)\le\|V\|_{L^1}O(1/k) \qquad \text{as } k\to\infty. \qquad (36)$$

Some simple computations, then imply the relation (28) and finish the proof of the Theorem 2. $\square$

V. Conclusion

As we have seen, the high-frequency asymptotic behavior of the reflection coefficient, over a fixed semi-infinite edge of a star-shape graph, must provide us very interesting and useful information on the geometry and the topology of our network. Two main directions may be considered for further extension of this result. The first direction deals with the more general case of trees instead of simply star-shape graphs. The second one corresponds to the case of LCRG transmission lines where one needs to extend all the utilities to the Zakharov-Shabat operator [8].

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